

Photonic Hall effect in cold atomic clouds

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Abstract: On the basis of exact numerical simulations and analytical calculations, we describe qualitatively and quantitatively the interference processes at the origin of the photonic Hall effect for resonant Rayleigh (point-dipole) scatterers in a magnetic field. For resonant incoming light, the induced giant magneto-optical effects result in relative Hall currents in the percent range, three orders of magnitude larger than with classical scatterers. This suggests that the observation of the photonic Hall effect in cold atomic vapors is within experimental reach.

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Light propagation in homogeneous media in the presence of a static magnetic field is a rich and vivid field of research where the symmetries dictated by nature lead to subtle magneto-optical phenomena [1]. About ten years ago, the question of magneto-optics in strongly scattering media was addressed and several effects bearing close analogies with electronic transport were theoretically predicted and observed [2]. One striking example is given by the so-called photonic Hall effect (PHE) where light propagating in a scattering medium subject to a transverse magnetic field can be deflected in the direction perpendicular to both the incident beam and the magnetic field [3, 4]. Cold atomic gases provide an appealing testing ground for such effects in the multiple scattering regime. Indeed they constitute a perfect monodisperse sample of highly-resonant point scatterers which are very sensitive to external perturbations and where spurious phase-breaking mechanisms can be easily circumvented. Typically few Gauss are enough to induce strong magneto-optical effects like the Faraday rotation [5] in sharp contrast with classical materials where Teslas are needed to induce significant effects. If the impact of a magnetic field on coherent backscattering has already been studied [6, 7], the question of the observation of the PHE in atomic vapors is still open. In this paper, we present analytical and numerical calculations identifying the physical origin of the PHE for point-dipole scatterers without internal degeneracies. Our results show that the effect should be observable in cold atomic gases.

For a quantitative study of the PHE, one needs to address the question of directional asymmetries displayed by the configuration-averaged radiation pattern of an assembly of atoms located at random positions and illuminated by an incident monochromatic plane wave (wave vector \mathbf{k} , polarization vector $\boldsymbol{\epsilon} \perp \mathbf{k}$, angular frequency $\omega = ck = 2\pi c/\lambda$) while being subjected to an external static magnetic field with strength B pointing in the direction $\hat{\mathbf{B}}$. We consider here the simplest possible

atomic internal structure serving our purposes, namely two-level atoms having a groundstate with angular momentum $J = 0$ connected by a narrow optical dipole transition to an excited state with angular momentum $J_e = 1$. The energy separation between the atomic states is $\hbar\omega_0 = \hbar c/\lambda_0$ and the natural energy width of the excited state is $\hbar\Gamma \ll \hbar\omega_0$. This is one of the best possible natural realizations of resonant point scatterers [8] and it corresponds for example to the case of ^{88}Sr atoms ($\lambda_0 = 461\text{nm}$, $\Gamma/2\pi = 32\text{MHz}$, Landé factor of the excited state $g_e = 1$). When $B = 0$, the incident light is quasi-resonant ($\lambda \approx \lambda_0$) with this optical dipole transition and we will denote by $\delta = (\omega - \omega_0)$ the light detuning with respect to the atomic line ($\delta \ll \omega_0$). In the rest of the paper, we assume that the incident light intensity is low enough to neglect all nonlinear effects. When the magnetic field is applied, the internal degeneracy is lifted (Zeeman effect) and the excited level is split into 3 components separated by μB where $\mu/2\pi = 1.4\text{MHz/G}$ is the Zeeman shift rate. As soon as the Zeeman shift becomes comparable to the resonance width, i.e $\phi_B = 2\mu B/\Gamma \sim 1$, the scattering properties of each atom are strongly modified (this would occur at $B \sim 11\text{G}$ in the case of ^{88}Sr).

The source of the field radiated by the atom is the oscillating electric dipole moment $\mathbf{d} \exp(-i\omega t)$ induced by the incident electric field $\mathbf{E} \exp(-i\omega t)$. The radiated spectrum is elastic because there is no Zeeman effect in the groundstate. The situation would be more involved for atoms with a degenerate groundstate where frequency changes are possible, leading to an inelastic spectrum. In our situation, $\mathbf{d} = \epsilon_0 \boldsymbol{\alpha}(\mathbf{B})\mathbf{E}$ and the radiation properties are fully characterized by the polarizability tensor $\boldsymbol{\alpha}(\mathbf{B}) = \alpha_0 \mathcal{T}(\mathbf{B})$ given by:

$$\alpha_0 = \frac{6\pi}{k^3} \frac{\Gamma/2}{\delta + i\Gamma/2} \quad (1)$$

$$\mathcal{T}(\mathbf{B}) = \zeta(B) \mathbb{1} + \eta(B) \mathbb{1} \times \hat{\mathbf{B}} + \xi(B) \hat{\mathbf{B}} \hat{\mathbf{B}}$$

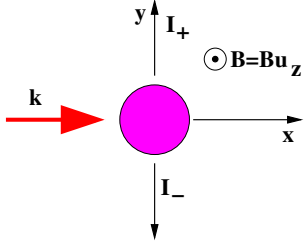


FIG. 1: (Color online) The photonic Hall geometry. A plane wave $\mathbf{k} = k\hat{\mathbf{x}}$ is scattered by a cloud of atoms subjected to a static magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$. The Hall current is measured either in the linear polarization channel $\hat{\mathbf{y}} \rightarrow \hat{\mathbf{x}}$ or in the $\hat{\mathbf{y}} \rightarrow \hat{\mathbf{z}}$ one. It is defined as $\Delta I = (I_+ - I_-)$, with I_{\pm} the configuration-averaged differential cross-section along $\pm\hat{\mathbf{y}}$.

where α_0 is the complex atomic polarizability at $B = 0$. The dyadic tensor $\mathcal{T}(\mathbf{B})$ embodies the effect of the magnetic field on the photon polarization degrees of freedom and gives rise to the usual magneto-optical effects. The ζ term is responsible for the normal extinction of the forward beam in a scattering medium made of such atoms (Lambert-Beer law). The η term describes the magnetically-induced rotation of the atomic dipole moment (Hanle effect) [6] and induces Faraday rotation and dichroism effects in the forward beam when $\mathbf{k} \parallel \mathbf{B}$ [1]. Finally, the ξ term is responsible for the Cotton-Mouton effect also observed in the forward beam when $\mathbf{k} \perp \mathbf{B}$ [1]. These coefficients read:

$$\zeta = \frac{1}{1+\phi^2}, \eta = \frac{\phi}{1+\phi^2}, \xi = \frac{\phi^2}{1+\phi^2}, \phi = \frac{\phi_B}{1-2i\delta/\Gamma}, \quad (2)$$

and are real on resonance ($\delta = 0$). From the polarizability we then get the single-atom differential cross-section:

$$I(\mathbf{k}\epsilon \rightarrow \mathbf{k}'\epsilon') = \frac{k^4}{16\pi^2} |\bar{\epsilon}' \underline{\alpha}(\mathbf{B}) \epsilon|^2 = \frac{3\sigma_0}{8\pi} |\bar{\epsilon}' \mathcal{T} \epsilon|^2 \quad (3)$$

where $\sigma_0 = |\alpha_0|^2 k^4 / 6\pi$ is the total scattering cross-section at zero magnetic field. As immediately seen, the single atom differential cross-section (3) only depends on the incoming and outgoing polarizations and is thus completely insensitive to the change $\mathbf{k}' \rightarrow -\mathbf{k}'$. For an isolated atom there is thus no possible asymmetry when reversing the direction of observation [9, 10]. As a consequence, the single scattering signal originating from an assembly of such atoms cannot display any directional asymmetry and the PHE, if any, must come from a multiple scattering effect.

Before considering the general case, we first analyze the radiation properties of two isolated atoms located at positions \mathbf{r}_1 and \mathbf{r}_2 . This is the simplest possible situation where multiple scattering plays a role and it will allow us to extract physical insights about possible mechanisms at work in the PHE. In the Hall geometry (see Fig. 1), one measures the differential cross sections I_{\pm} along the up and down directions $\pm\hat{\mathbf{y}}$ perpendicular both to the

incident light direction $\mathbf{k} = k\hat{\mathbf{x}}$ and to the magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$. The total radiation field is the sum of the fields radiated by the atomic dipoles induced by the incoming and scattered fields at their respective positions. The exact solution thus involves the inversion of a linear system of 2 coupled vectorial equations where the polarizability tensor plays a key role. For a fixed relative distance $r = |\mathbf{r}_1 - \mathbf{r}_2|$ between the atoms, we compute exactly the differential cross sections I_{\pm} and average them over all possible relative orientations of the atoms to get $\langle I_{\pm} \rangle$. We then extract the Hall current $\Delta I = \langle I_+ \rangle - \langle I_- \rangle$, the mean intensity $I = (\langle I_+ \rangle + \langle I_- \rangle)/2$ and the relative Hall current $\beta = \Delta I/I$. There are 4 possible linear polarization channels $\epsilon \rightarrow \epsilon'$ for the data analysis. However we first note (and we have numerically checked) that the Hall currents *must be the same* in the linear channels $\hat{\mathbf{y}} \rightarrow \hat{\mathbf{z}}$ and $\hat{\mathbf{z}} \rightarrow \hat{\mathbf{x}}$ since they are related by time-reversal symmetry. Second, as we also checked numerically, the Hall current *must vanish* in the $\hat{\mathbf{z}} \parallel \hat{\mathbf{z}}$ channel since the polarizations being along the magnetic field, they are insensitive to the Hanle effect. We are thus left with the two *lin* \perp *lin* channels $\hat{\mathbf{y}} \rightarrow \hat{\mathbf{x}}$ and $\hat{\mathbf{y}} \rightarrow \hat{\mathbf{z}}$. Fig. 2 summarizes our numerical results. We see that the Hall current vanishes as $kr \rightarrow 0$. Indeed for very small distances, the two radiating dipoles are always in phase and thus add up constructively meaning that the two atoms behave like a single scatterer with a dipole moment twice larger, a situation for which we already know there is no possible directional asymmetry. We also see that for low ϕ values, the relative Hall current decreases in the $\hat{\mathbf{y}} \rightarrow \hat{\mathbf{z}}$ channel, whereas, in the $\hat{\mathbf{y}} \rightarrow \hat{\mathbf{x}}$ channel, it is comparable to the one for high ϕ values. The reason is that, in the $\hat{\mathbf{y}} \rightarrow \hat{\mathbf{x}}$ channel, the single scattering due to the Hanle effect increases with the magnetic field, such that the background intensity in this channel gets more and more contaminated (and even dominated at large distances) by the single scattering signal. In the $\hat{\mathbf{y}} \rightarrow \hat{\mathbf{z}}$ channel, it is always filtered out.

To get some insights about the physical processes at work, we consider the case of a “dilute” medium $kr \gg 1$ and we expand the field radiated by the two atoms in powers of $(kr)^{-1}$. Skipping tedious details, at leading order the scattered amplitude is obtained from the sum of the two diagrams shown in Fig. 3a, the respective amplitude being denoted by u and v . Each differential cross-section I_{\pm} contains thus interference terms (i.e. $u\bar{v} + v\bar{u}$) and background terms (i.e. $|u|^2 + |v|^2$), but since the later are independent of the scattering direction, they cancel out in the difference. The Hall current is thus solely given by a difference of interference terms, which are precisely the crossed terms at the heart of the coherent backscattering effect [11]. More precisely, the Hall current reads

$$\Delta I = \langle \delta I(\mathbf{r}, \mathbf{B}) \{ \cos[(\mathbf{k} + \mathbf{k}') \cdot \mathbf{r}] - \cos[(\mathbf{k} - \mathbf{k}') \cdot \mathbf{r}] \} \rangle, \quad (4)$$

where $\delta I(\mathbf{r}, \mathbf{B}) \propto |\bar{\epsilon}' \mathcal{T}(\phi) \underline{\Delta}_r \mathcal{T}(\phi) \epsilon|^2$, $\langle \dots \rangle$ denotes the

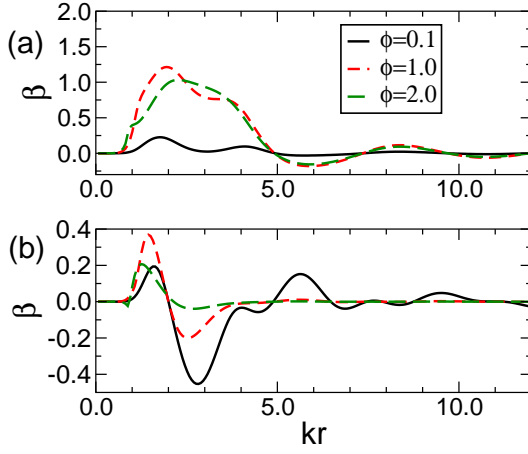


FIG. 2: (Color online) Two-atom case. Relative Hall current β observed at resonance $\delta = 0$ in the linear polarization channels $\hat{y} \rightarrow \hat{x}$ (a) and $\hat{y} \rightarrow \hat{z}$ (b) as a function of the relative distance kr for different values of $\phi_B = 2\mu_B/\Gamma$.

average over the relative orientation of the two atoms and $\Delta_{\mathbf{r}}$ is the projector onto the plane perpendicular to \mathbf{r} . After elementary manipulations, equation (4) can be conveniently rewritten as follows:

$$\Delta I = \langle (\delta I(\mathbf{r}, \mathbf{B}) - \delta I(\mathbf{r}, -\mathbf{B})) \cos[(\mathbf{k} + \mathbf{k}') \cdot \mathbf{r}] \rangle. \quad (5)$$

The Hall current can then be understood as a difference between the two configuration-averaged interference effects generated in the *same* direction \mathbf{k}' , but for opposite directions of the magnetic field. The imbalance results from the interplay between the dipole rotation induced by the Hanle effect and the transverse projector $\Delta_{\mathbf{r}}$. Moreover, since the cosine term depicts very fast oscillations, a stationary phase approximation shows that the main contribution in Eq. (5) comes from the configurations where the two atoms are aligned along the $\mathbf{k} + \mathbf{k}'$ direction.

Performing the angular average, one gets the Hall current in the $\hat{y} \rightarrow \hat{z}$ channel:

$$\Delta I \approx \frac{81}{8k^2} \frac{\phi_B}{|(1 - 2i\delta/\Gamma)^2 + \phi_B^2|^2} \frac{\cos(\sqrt{2}kr)}{(kr)^4}, \quad (6)$$

the $\sqrt{2}kr$ term simply corresponding to the value of $(\mathbf{k} + \mathbf{k}') \cdot \mathbf{r}$ when these two vectors are parallel to each other.

In the $\hat{y} \rightarrow \hat{x}$ channel, a closer inspection shows that, at same order, there is an additional contribution due to diagrams accounting for the interference between recurrent and single scattering processes (see Fig. 3b). The final expression for the Hall current in this channel is quite tedious but simplifies at resonance $\delta = 0$:

$$\Delta I \approx \frac{81\sqrt{2}}{8k^2} \frac{\phi_B}{(1 + \phi_B^2)^3} \frac{\sin(\sqrt{2}kr)}{(kr)^3} (1 + \cos(2kr)). \quad (7)$$

As shown in Fig. 4 for $\delta = 0$ and $\phi_B = 1$, the agreement between the numerical computations and these asymptotic results proves excellent as soon as $kr \gtrsim 10$. In

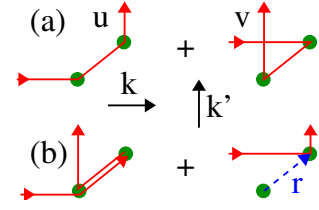


FIG. 3: (Color online) Two-atom case. For large distances, the PHE in the $\hat{y} \rightarrow \hat{z}$ channel results from the interference between the two diagrams (a). In the $\hat{y} \rightarrow \hat{x}$ channel an additional contribution comes from the interference between the two diagrams accounting for recurrent scattering (b) (and also between the ones obtained by exchanging the two atoms).

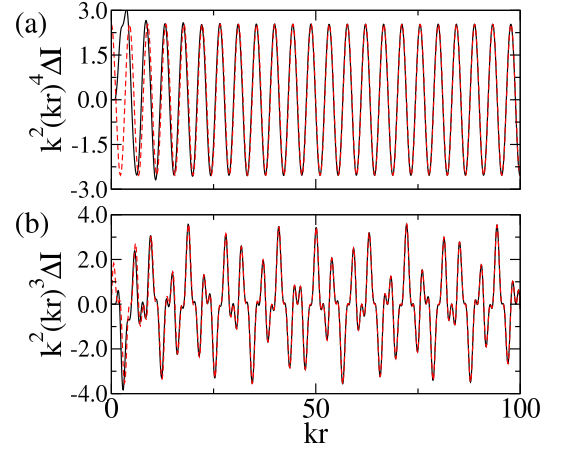


FIG. 4: (Color online). Normalized Hall current in the two-atom case. Comparison between the numerical calculations (plain curves) and the analytical predictions Eqs. (6),(7) (dashed curves) as a function of kr for $\delta = 0$ and $\phi_B = 1$. (a) $\hat{y} \rightarrow \hat{z}$ channel. (b) $\hat{y} \rightarrow \hat{x}$ channel. The excellent agreement emphasizes that the underlying physics of the PHE is fully caught by the diagrams depicted in Fig. 3.

particular it is quite remarkable that recurrent scattering is essential to reproduce the additional oscillations observed in the $\hat{y} \rightarrow \hat{x}$ channel and due to the $\cos 2kr$ term in Eq. (7). We have also numerically checked that the agreement is still excellent in both channels when $\delta \neq 0$. This gives clear evidence that the physical mechanism at the heart of the PHE for atomic scatterers is the interference between the scattering processes depicted by the diagrams in Fig. 3. In addition, the fact that, in the small ϕ_B limit, the Hall current is proportional to ϕ_B (i.e. to the atomic dipole rotation) actually reflects that only one active scatterer would be enough to generate a Hall current. Finally, one must also mention that, contrary to the case studied in Ref. [3], both our analytical and numerical results show that there is still a PHE when the antisymmetric part of the self-energy ($\Sigma \propto \underline{\alpha}(\mathbf{B})$) has a vanishing real part (i.e. at resonance $\delta = 0$), the exact dependence on δ being a smooth bell-shaped curve.

Assuming an isotropic propagation in the medium

TABLE I: The left side displays the relative Hall currents β at $\delta = 0$ and $\phi_B = 1$, in the Hall geometry, for 500 atoms uniformly distributed inside a sphere with optical density $\rho \approx 0.65$. The number of disorder configurations is 6×10^5 . The expected accuracy of the order of 10^{-3} is confirmed when comparing to the relative currents β (see right side) in the $\mathbf{k} \parallel \mathbf{B}$ configuration, when no PHE should show up.

$\mathbf{k} \perp \mathbf{B}$		$\mathbf{k} \parallel \mathbf{B}$	
$\hat{\mathbf{z}} \rightarrow \hat{\mathbf{x}}$	0.0579	$\hat{\mathbf{x}} \rightarrow \hat{\mathbf{z}}$	0.00267
$\hat{\mathbf{y}} \rightarrow \hat{\mathbf{z}}$	0.0545	$\hat{\mathbf{x}} \rightarrow \hat{\mathbf{x}}$	-0.00014
$\hat{\mathbf{y}} \rightarrow \hat{\mathbf{x}}$	-0.0088	$\hat{\mathbf{y}} \rightarrow \hat{\mathbf{z}}$	-0.00121
$\hat{\mathbf{z}} \rightarrow \hat{\mathbf{z}}$	0.0022	$\hat{\mathbf{y}} \rightarrow \hat{\mathbf{x}}$	-0.00004

(with a mean free path ℓ), one can further average the preceding expressions over all possible positions of the two scatterers. As seen from Eqs. (6) and (7), the interference effect leads to fast decreasing oscillations at the wavelength scale. Since $\lambda \ll \ell$, the main contribution to the Hall effect will arise for scatterers separated by $r \sim \lambda$. The probability of finding two such scatterers scales with the optical density $\rho = n\lambda^3$, n being the scatterer density; we thus expect the actual relative Hall current β to be smaller by a factor $(k\ell)^{-1}$ than the average background [12]. Obviously, the exact result will depend on the geometry of the medium and on the optical thickness. Nevertheless, the PHE in a assembly of resonant point scatterers, even if rather small, should be measurable.

To bring a numerical confirmation of our findings, we have considered $N = 500$ atoms uniformly distributed inside a sphere at an optical density $\rho \approx 0.65$ and illuminated by a plane wave set on resonance. This leads to $k\ell = 4\pi^2/3\rho \approx 20$ at $B = 0$. Such a value is difficult to achieve in a real experiment but has already been obtained [13]. The optical thickness along a diameter of the sphere is $b \approx 3.5$. The magnetic field value has been set at $\phi_B = 1$. To obtain the total radiated field, we have solved the corresponding system of $3N$ linear equations [14] and we have computed the various quantities of interest averaged over 6×10^5 configurations, such that we expect an accuracy of the order of 10^{-3} . To stress the existence of the PHE, we have also computed β in the geometry $\mathbf{k} \parallel \mathbf{B}$ where no PHE should be observed; as depicted by the right side of table I, the corresponding values are effectively at most of the order of few 10^{-3} . Up to this accuracy, the numerical results in the geometry $\mathbf{k} \perp \mathbf{B}$, are in a qualitative agreement with the two-atom case: there is no Hall current in the $\hat{\mathbf{z}} \rightarrow \hat{\mathbf{z}}$ channel. It is about the same order of magnitude in the two conjugate channels $\hat{\mathbf{y}} \rightarrow \hat{\mathbf{z}}$ and $\hat{\mathbf{z}} \rightarrow \hat{\mathbf{x}}$ ($\beta \approx 5.5\%$). Finally, it is larger in these channels than in the $\hat{\mathbf{y}} \rightarrow \hat{\mathbf{x}}$ channel ($|\beta| \approx 1\%$). To enforce the validity of our mesoscopic description, we have computed β for different values of $k\ell$, while keeping fixed the optical thickness b . The results are displayed in table II and, as expected, the product $\beta k\ell$ is, within the

TABLE II: Relative Hall current β in the $\hat{\mathbf{z}} \rightarrow \hat{\mathbf{x}}$ and $\hat{\mathbf{y}} \rightarrow \hat{\mathbf{z}}$ channels at a fixed optical thickness $b = 1.3$. Within the numerical accuracy, the results indicate that β scales like $1/(k\ell)$.

$k\ell$	40.00	63.25	89.44
$\beta \times k\ell$	4.0(1)	3.8(3)	4.0(3)

statistical errors, independent of $k\ell$. Finally, one must note that the values found here are in the percent range and are much larger (by three orders of magnitude) than observed with classical scatterers [4] although the magnetic field is smaller by two or three orders of magnitude. This arises because the Zeeman effect in highly-resonant atomic scatterers induces a "giant" dipole rotation which amplifies the PHE.

In summary, on the basis of numerical and analytical calculation, we have qualitatively and quantitatively explained the underlying interference effect at the origin of the photonic Hall effect for resonant point-dipole scatterers. Our results show that the effect, albeit small, should be observable in cold atomic vapors. Possible extensions of this work would consist first in developing the diagrammatic analysis to an arbitrary number of scattering events and, second, in accounting for internal degeneracies in the atomic groundstate. This would give a quantitative and comprehensive description of the photonic Hall effect in cold atomic clouds.

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